

ON SENSITIVITY TO INITIAL CONDITIONS AND UNIQUENESS OF CONJUGACIES FOR STRUCTURALLY STABLE DIFFEOMORPHISMS

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ABSTRACT. In this paper we study C^1 -structurally stable diffeomorphisms, that is, C^1 Axiom A diffeomorphisms with the strong transversality condition. In contrast to the case of dynamics restricted to a hyperbolic basic piece, structurally stable diffeomorphisms are in general not expansive and the conjugacies between C^1 -close structurally stable diffeomorphisms may be non-unique, even if there are assumed C^0 -close to the identity. Here we give a necessary and sufficient condition for a structurally stable diffeomorphism to admit a dense subset of points with expansiveness and sensitivity to initial conditions. Moreover, we prove that the set of conjugacies between elements in the same conjugacy class is homeomorphic to the C^0 -centralizer of the dynamics. Finally, we use this fact to deduce that any two C^1 -close structurally stable diffeomorphisms are conjugated by a unique conjugacy C^0 -close to the identity if and only if these are Anosov.

1. INTRODUCTION

One of the leading problems considered by the dynamical systems community has been to provide a global view of the space of dynamical systems. In fact, based on the pioneering works of Andronov, Pontryagin, Peixoto and Smale, in the nineties Palis proposed a conjecture that roughly describes the complement of uniform hyperbolicity as the space of diffeomorphisms that are approximated by those exhibiting either homoclinic tangencies or heteroclinic cycles. This program has been carried out with success in the C^1 -topology, where perturbation tools as the Pugh closing lemma, the Franks' lemma, the Hayashi's connecting lemma or the Mañé's ergodic closing lemma are available (see e.g. [18, 14, 10] and references therein). Uniform hyperbolicity helped to coin the idea of stability that is, to characterize the dynamics that persist and behave similarly under small perturbations. The stability theorem for isolated hyperbolic sets asserts that any isolated hyperbolic basic set Λ for a C^1 -diffeomorphism f admits a continuation Λ_g for any C^1 -small perturbation g of the original dynamics, whose dynamics is topologically conjugate to $f|_{\Lambda_f}$: there exists a unique homeomorphism $h_g : \Lambda_f \rightarrow \Lambda_g$ that is C^0 -close to the identity and so that $h_g \circ f = g \circ h_g$. The uniqueness of the conjugacy h_g at finite distance to the identity reflects the rigidity of hyperbolicity and can be obtained, via Banach fixed point theorem, as a fixed point for the operator $L(h) = g^{-1} \circ h \circ f$ acting on the space $\text{Homeo}(M)$ of homeomorphisms in a compact manifold M .

A question that arises naturally in the vein of the stability for uniformly hyperbolic dynamics is to understand if hyperbolicity is a necessary and sufficient condition to characterize structurally stable diffeomorphisms. Recall that a C^1 -diffeomorphism f is called *structurally stable* if there exists an open neighborhood \mathcal{U} of f in the C^1 -topology such that for any $g \in \mathcal{U}$ there exists a homeomorphism $h_g : M \rightarrow M$ such that h_g conjugates the dynamics, that is, $h_g \circ f = g \circ h_g$. After the first examples of Ω -explosions there was a clear idea that heteroclinic cycles and tangencies constitute obstructions to structural stability and that uniform hyperbolicity should play a key role (see e.g. [26]). Robbin, Robinson and Mañé [22, 23, 14] completed the proof that C^1 -structural stability is equivalent to the Axiom A and the strong transversality conditions (in dimension two this was obtained by de Melo [5]). In particular, if f is structurally stable, then there exists a Df -invariant decomposition $T_{\Omega(f)}M = E^s \oplus E^u$ and constants $C > 0$ and $\lambda \in (0, 1)$ such that

$$\|Df^n(x)|_{E_x^s}\| \leq C\lambda^n \quad \text{and} \quad \|Df^{-n}(x)|_{E_x^u}\| \leq C\lambda^n$$

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for every $n \geq 1$ and every x in the non-wandering set $\Omega(f)$, and $\Omega(f)$ coincides with the closure of the periodic points. Moreover, the non-wandering set of a structurally stable diffeomorphism can be decomposed as a finite set of hyperbolic sets whose basins of attraction cover the entire manifold. A priori, such decomposition could suggest that the rigidity of the conjugacies on each hyperbolic basic set would ‘spread’ to the manifold or, in other words, that the global conjugacy could be completely determined by the conjugacies of each of the hyperbolic basic pieces.

Our purpose here is to revisit structurally stable diffeomorphisms, discussing both expansiveness properties as their space of conjugacies given, as starting point, the (local) Ω -stability of hyperbolic basic pieces. In the context of structural stability the notion of expansiveness is associated to a very rigid phenomenon, because a C^1 -structurally stable diffeomorphism is expansive if and only if it is Anosov [15]. In the context of compact surfaces, all expansive homeomorphisms on surfaces are conjugate to Anosov diffeomorphisms if $M = \mathbb{T}^2$ and are pseudo-Anosov if the genus is larger or equal to two [12]. Thus, \mathbb{S}^2 admits structurally stable diffeomorphisms but admits no expansive homeomorphisms. In view of the later result it is natural to ask whether subsets of points with some expansiveness can be topologically large. Although there are structurally stable diffeomorphisms that admit no dense subset of points with expansiveness (see e.g. Example 7.1) we prove that all structurally stable diffeomorphisms whose topological basins of trivial attractors and repellers in $\Omega(f)$ do not intersect admit a dense subset of points with expansiveness (cf. Theorem 1). In rough terms, dense expansiveness holds if and only if some north-pole south-pole kind of dynamics cannot be embedded in the original dynamical system, and this condition implies on the sensitivity to initial conditions.

The second part of our work concerns the study of the set of conjugacies appearing naturally in the context of structural stability. If f is a structurally stable diffeomorphism there exists an open neighborhood \mathcal{U} of f in the C^1 -topology such that for any $g \in \mathcal{U}$ there exists a homeomorphism $h_g : M \rightarrow M$ such that h_g conjugates both dynamics, that is, $h_g \circ f = g \circ h_g$. It is natural to ask under which conditions this conjugacy can be taken unique C^0 -close to the identity. Our approach to this problem is to relate the set of all such conjugacies with the C^0 -centralizer of the dynamics by proving that for any diffeomorphism g in the conjugacy class of a given diffeomorphism f the space $\mathcal{H}_{f,g}$ of conjugacies between f and g is homeomorphic to the C^0 -centralizer $Z^0(f) = \{h \in \text{Homeo}(M) : h \circ f = f \circ h\}$ of f (cf. Theorem 2). This leads to the comprehension of the C^0 -centralizer of C^1 -structurally stable diffeomorphisms. In his seminal paper [27], Smale conjectured that ‘most’ dynamical systems would have trivial C^1 -centralizer. Although this is not yet completely understood, there are evidences that Smale’s conjecture is true and some important contributions in that direction include e.g. [13, 17, 4, 2, 6, 1]. From the purely topological viewpoint, one cannot expect the conjecture of Smale to hold. Indeed, C^0 -centralizers are larger and there are open sets of surface C^1 -Anosov diffeomorphisms whose C^0 -centralizer is discrete but non-trivial [24]. We refer the reader to Section 2 for precise definitions and a more detailed discussion. We include some examples (see Examples 7.1 and 7.2) to illustrate that even in the absence of trivial basic pieces of the non-wandering set the conjugacies may contain a continuum of homeomorphisms C^0 -close to the identity. We prove that the C^0 -centralizer of a structurally stable diffeomorphism is discrete if and only if it is an Anosov diffeomorphism (cf. Theorem 3). As a consequence, we deduce Mañé characterization is optimal: there exist structurally stable diffeomorphisms with an invariant and dense set of points with expansiveness which are not Anosov (cf. Example 7.2).

The paper is organized as follows. In Section 2 we recall the notions and some results about uniformly hyperbolic sets, homoclinic classes and structural stability. In Section 3 we state our main results. The proofs of the two main results will appear from Sections 4 to 6. Section 7 is devoted to examples.

2. PRELIMINARIES

In this section we recall some results about uniformly hyperbolic sets, homoclinic classes and structurally stable diffeomorphisms. Throughout, let M be a compact Riemannian manifold, let d denote the Riemannian distance on M , and let $\text{Diff}^r(M)$ ($r \geq 1$) denote the space of C^r -diffeomorphisms on M .

Hyperbolic sets, expansiveness and sensitivity to initial conditions. Given $f \in \text{Diff}^r(M)$ ($r \geq 1$), let $\text{Per}(f)$ denote the set of periodic points for f and let $\Omega(f) \subset M$ denote the non-wandering set of f . We say that a

compact f -invariant set $\Lambda \subset M$ is a *uniformly hyperbolic* for f if there exists a Df -invariant splitting $T_\Lambda M = E^s \oplus E^u$ and constants $C > 0$ and $\lambda \in (0, 1)$ so that $\|Df^n(x)|_{E_x^s}\| \leq C\lambda^n$ and $\|(Df^n(x)|_{E_x^u})^{-1}\| \leq C\lambda^n$ for every $x \in \Lambda$ and $n \geq 1$. A periodic point $p \in \text{Per}(f)$ is hyperbolic if $O(p) = \{p, f(p), \dots, f^{n-1}(p)\}$ is a hyperbolic set for f , where n stands for the period of p . Given an f -invariant set Λ , we say that $f|_\Lambda$ is *transitive* if it has a dense orbit, that is, there exists $x \in \Lambda$ so that $O(x) := \{f^n(x) : n \in \mathbb{Z}\}$ is dense in Λ . Given a point $x \in M$ and $\varepsilon > 0$ the ε -stable set of x is defined by $W_\varepsilon^s(x) = \{y \in M : d(f^n(y), f^n(x)) \leq \varepsilon \text{ for all } n \geq 0\}$. Similarly, $W_\varepsilon^u(x) = \{y \in M : d(f^{-n}(y), f^{-n}(x)) \leq \varepsilon \text{ for all } n \geq 0\}$ is the ε -unstable set of x . Given a hyperbolic set Λ for f there exists a uniform $\varepsilon > 0$ so that the stable and unstable sets $W_\varepsilon^s(x)$ and $W_\varepsilon^u(x)$ are C^r submanifolds tangent to E_x^s and E_x^u , respectively, for every $x \in \Lambda$. These are called, respectively, the *local stable* and *local unstable manifolds* at x of size ε . A subset $\Delta \subset W^s(x)$ is called a *fundamental domain* if for any $z \in W^s(x) \setminus \{x\}$ there exists a unique $n = n(z) \in \mathbb{Z}$ so that $f^n(z) \in \Delta$. We refer the reader to [25] for more details.

Given $\varepsilon > 0$ we say that $\Lambda \subset M$ is a set of ε -expansiveness for f if for any $x \neq y \in \Lambda$ there exists $n \in \mathbb{Z}$ so that $d(f^n(x), f^n(y)) > \varepsilon$. Observe that no f -invariance condition is required. When no confusion is possible we will refer to these simply as sets of expansiveness. We say that f is *densely expansive* if there exists an ε -expansive dense set in M , for some $\varepsilon > 0$. An homeomorphism f is *expansive* if the manifold M is a set of expansiveness for f . Expansive homeomorphisms in compact manifolds of dimension smaller or equal to 2 are either Anosov or pseudo-Anosov (see e.g. [12, 11] and references therein). Thus, there are geometrical and topological obstructions for a manifold to admit expansive dynamics. To the best of our knowledge there is still no complete classification of expansive homeomorphisms in dimension larger than 2.

Finally we recall the notion of sensitivity to initial conditions. We say that f has *sensitivity to initial conditions* if there exists $\varepsilon > 0$ so that every $x \in M$ is accumulated by a sequence of points $(x_n)_n$ in M such that $\sup_{k \in \mathbb{Z}} d(f^k(x), f^k(x_n)) > \varepsilon$. This condition, weaker than expansiveness does not hold for all structurally stable diffeomorphisms (e.g. north-pole south-pole dynamics).

C^r -centralizers. Given $f \in \text{Diff}^r(M)$, $r \in \mathbb{N} \cup \{\infty\}$ and $0 \leq k \leq r$, the C^k -centralizer for f is the set $Z^k(f) = \{g \in \text{Diff}^k(M) : g \circ f = f \circ g\}$, where $\text{Diff}^0(M)$ stands for the space $\text{Homeo}(M)$. For every $0 \leq k \leq n$, it is clear that $Z^k(f)$ is a subgroup of $(\text{Diff}^k(M), \circ)$ and it contains the subgroup $\{f^n : n \in \mathbb{Z}\}$. Clearly, the following inclusion holds $Z^0(f) \supset Z^1(f) \supset Z^2(f) \supset \dots \supset Z^r(f) \supset \{f^n : n \in \mathbb{Z}\}$. We say that $f \in \text{Diff}^r(M)$ has *trivial C^k -centralizer* if the centralizer is the smallest possible, that is, $Z^k(f) = \{f^n : n \in \mathbb{Z}\}$. We say that the diffeomorphism $f \in \text{Diff}^r(M)$ has *discrete C^k -centralizer* if there is $\varepsilon > 0$ so that $d_0(h_1, h_2) > \varepsilon$ for every distinct $h_1, h_2 \in Z^k(f)$. Walters [28] proved that expansive homeomorphisms have discrete C^0 -centralizers.

Axiom A diffeomorphisms and homoclinic classes. In what follows we collect some results on Axiom A diffeomorphisms (proofs and more details can be found in [25]). We say that a diffeomorphism $f \in \text{Diff}^r(M)$, $r \geq 1$, is *Axiom A* if (i) $\overline{\text{Per}(f)} = \Omega(f)$ and (ii) $\Omega(f)$ is a uniformly hyperbolic set. Clearly all periodic points of Axiom A diffeomorphisms are hyperbolic. We say that f is an *Anosov diffeomorphism* if the whole manifold M is a hyperbolic set for f . If $p \in \text{Per}(f)$ is a hyperbolic periodic point for f of period $k \geq 1$ then there exists a C^1 -neighborhood \mathcal{U} of f and a neighborhood $V \subset M$ of p so that any $g \in \mathcal{U}$ admits a unique hyperbolic periodic point $p(g) \in V \cap \text{Per}(g)$ of period k , referred as the *continuation* of p . Given a hyperbolic periodic point p , the *homoclinic class* $H(p, f)$ for p with respect to f is defined by $H(p, f) = \overline{W^u(O(p))} \cap \overline{W^s(O(p))}$. By the spectral decomposition theorem, for any Axiom A diffeomorphism f there are finitely many (hyperbolic) periodic points p_ℓ ($1 \leq \ell \leq k$) and hyperbolic homoclinic classes $\Lambda_\ell = H(p_\ell, f)$ so that $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \dots \cup \Lambda_k$ (*spectral decomposition*). For every C^1 -small perturbation g of f the hyperbolic set $H(p_\ell, f)$ admits a continuation $H(p_\ell(g), g)$. Consider the partial order on the space $\{\Lambda_i : 1 \leq i \leq k\}$ given by $\Lambda_i > \Lambda_j$ if and only if $W^u(\Lambda_i) \cap W^s(\Lambda_j) \neq \emptyset$. We say that f has *no cycles* if whenever $\Lambda_i > \Lambda_j$ and $\Lambda_j > \Lambda_i$ we have that $i = j$. Moreover, if f is Axiom A with no cycles then there exists a *filtration* adapted to $\Omega(f)$: there exists a nested sequence $\emptyset = M_0 \subset M_1 \subset \dots \subset M_k = M$ of smooth codimension 0 submanifolds with boundary so that $f(M_i) \subset \text{int}(M_i)$ and $\bigcap_{n \in \mathbb{Z}} f^n(M_i \setminus M_{i-1})$ consists of a finite union of basic pieces of $\Omega(f)$ for every $1 \leq i \leq k$. We say that an f -invariant, compact and transitive set Λ is an *attractor* if there exists an open neighborhood U of Λ so that $\Lambda = \bigcap_{n \geq 0} f^n(\overline{U})$. An f -invariant set is a *repeller* if it is an attractor for f^{-1} . Finally we say that a basic piece of the non-wandering set is trivial if it consists of a periodic orbit. Given an attractor Λ for f , the

topological basin of attraction is the set $B(\Lambda) = \{x \in M : \text{dist}(f^n x, \Lambda) \rightarrow 0 \text{ as } n \rightarrow \infty\}$. The topological basin of repulsion for some repeller Λ is defined as the topological basin of attraction for Λ with respect to f^{-1} .

Conjugacy classes and structural stability. Given a C^r -diffeomorphism f , $r \geq 1$, we define the *conjugacy class* of the diffeomorphism f as the set $C_f = \{g \in \text{Diff}^r(M) : \text{there exists } h \in \text{Homeo}(M) \text{ s.t. } h \circ g = f \circ h\}$. Consider $\mathcal{H}_f = \bigcup_{g \in C_f} \mathcal{H}_{f,g}$ where $\mathcal{H}_{f,g} = \{h \in \text{Homeo}(M) : h \circ f = g \circ h\}$. In other words, $\mathcal{H}_{f,g}$ denotes the space of all homeomorphisms that conjugate f and g . Endow the space of homeomorphisms $\text{Homeo}(M)$ on M with the distance d_0 defined by

$$d_0(h_1, h_2) = \sup_{x \in M} d(h_1(x), h_2(x)) + \sup_{x \in M} d(h_1^{-1}(x), h_2^{-1}(x))$$

for all $h_1, h_2 \in \text{Homeo}(M)$. Given $f \in \text{Diff}^1(M)$, we say that f is *structurally stable* if there exists an open neighborhood \mathcal{U} of f in the C^1 -topology such that for any $g \in \mathcal{U}$ there exists a homeomorphism $h_g : M \rightarrow M$ such that h_g conjugates the dynamics, that is, $h_g \circ f = g \circ h_g$. Robbin, Robinson and Mañé [21, 23, 14] proved that a C^1 -diffeomorphism is structurally stable if and only if it is Axiom A and satisfies the *strong transversality condition*: if $E_x^\pm = \{v \in T_x M : \lim_{n \rightarrow \infty} \|Df^{\mp n}(x)v\| = 0\}$ then $T_x M = E_x^- + E_x^+$ for every $x \in M$. Moreover, if f is an Axiom A diffeomorphism then the strong transversality condition is equivalent to the transversal intersection of every stable and unstable manifolds [21, Proposition 7.5]. In the proof of the first part of the stability conjecture, Robbin [21], Robinson [23] used the strategy developed by Moser [16] (in the proof of the stability of Anosov diffeomorphisms) to construct conjugacies that vary continuously with the dynamical system:

Theorem 2.1. (Robbin, Robinson) *Let f be a C^1 Axiom A diffeomorphism with the strong transversality condition. Then there exists an open neighborhood \mathcal{U} of f and $K > 0$ such that, for any $g \in \mathcal{U}$:*

- (i) *there exists $h = h_g \in \text{Homeo}(M)$ so that $h_g \circ f = g \circ h_g$; and*
- (ii) *$d_{C^0}(h_g, \text{id}) \leq K d_{C^0}(f, g)$.*

The proof of the previous theorem relies on Banach's fixed point theorem for a family of operators \mathcal{L}_g that vary continuous with the diffeomorphism g in a C^1 -neighborhood \mathcal{U} of f , but whose construction depends on some fixed neighborhoods of the basic sets and a partition of unity for M (cf. [21]). For that reason the selected conjugacies can be chosen to depend continuously with respect to the dynamics but uniqueness is not guaranteed. In fact, uniqueness of conjugacies C^0 -close to the identity may fail (cf. Example 7.1). This is in a strong contrast with the fact that the conjugacy restricted to each basic piece of the non-wandering set is unique and Hölder continuous. Finally, for completeness we observe that if $f \in \text{Diff}^2(M)$ is C^1 -structurally stable then there exists a C^1 -open neighborhood \mathcal{U} of f so that the conjugacy map $\mathcal{U} \ni g \mapsto h_g \in \text{Homeo}(M)$ can be chosen to be C^1 -differentiable [7, Theorem 2].

3. STATEMENT OF THE MAIN RESULTS

This section is devoted to the statement of our main results.

Expansiveness. First we provide a necessary and sufficient condition for a structurally stable diffeomorphism to be densely expansive.

Theorem 1. *Let $f \in \text{Diff}^1(M)$ be a structurally stable diffeomorphism. The following are equivalent:*

- (1) *the topological basins of trivial attractors and repellers in $\Omega(f)$ do not intersect;*
- (2) *f is densely expansive (i.e. there exists $\varepsilon > 0$ and a dense subset $D \subset M$ such that for any $x \neq y \in D$ there exists $n \in \mathbb{Z}$ satisfying $d(f^n(x), f^n(y)) > \varepsilon$);*
- (3) *f has sensitivity to initial conditions.*

One should mention that whenever the topological basins of trivial attractors and repellers in $\Omega(f)$ do not intersect, in the proof of Theorem 1, we construct a dense set of expansiveness $D \subset M$ which is not f -invariant and its saturated set $\bigcup_{n \in \mathbb{Z}} f^n(D)$ is not necessarily expansive.

Corollary 1. *Every structurally stable diffeomorphism whose the topological basins of trivial attractors and repellers in $\Omega(f)$ do not intersect is sensitive to initial conditions.*

Conjugacy classes and C^0 -centralizers. The following simple result builds a bridge between the C^0 -centralizer of a dynamical system f and set of conjugacies between diffeomorphisms in the same conjugacy class.

Theorem 2. *Let $f \in \text{Diff}^r(M)$, $r \geq 0$. For every $g \in C_f$ and $h \in \mathcal{H}_{f,g}$ the map*

$$F_h: \begin{array}{ccc} Z^0(f) & \rightarrow & \mathcal{H}_{f,g} \\ \tilde{f} & \mapsto & h \circ \tilde{f} \end{array}$$

is an homeomorphism and satisfies $F_h(\tilde{f} \circ \hat{f}) = F_h(\tilde{f}) \circ h^{-1} \circ F_h(\hat{f})$ for every $\tilde{f}, \hat{f} \in Z^0(f)$. In particular,

- (i) $\mathcal{H}_{f,g}$ is homeomorphic to $Z^0(f)$ for every $g \in C_f$;
- (ii) \mathcal{H}_f is homeomorphic to $C_f \times Z^0(f)$;
- (iii) $\mathcal{H}_{f,g}$ is a discrete subset of $\text{Homeo}(M)$ if and only if $Z^0(f)$ is a discrete subgroup of $\text{Homeo}(M)$.

Since the map F_h is an homeomorphism then the cardinality and topological properties of all sets $\mathcal{H}_{f,g}$ coincide for every $g \in C_f$. This motivates the following definition. Given $r \geq 0$, let $N^0(f) \in \mathbb{N} \cup \{\infty\}$ denote the minimum number of generators for the subgroup $Z^0(f)$. We are interested in studying the regularity of the following function:

$$N^0: \begin{array}{ccc} \text{Diff}^r(M) & \rightarrow & \mathbb{N} \cup \{\infty\} \\ f & \mapsto & N^0(f) \end{array}$$

The following is an immediate consequence of Theorem 2:

Corollary 2. *$N^0(\cdot)$ is a topological invariant: if f and g are topologically conjugate then $N^0(f) = N^0(g)$. In particular, $N^0(\cdot)$ is a locally constant function restricted to the space of structurally stable diffeomorphisms.*

As one could expect, the minimal cardinality $N^1(\cdot)$ of generators for the C^1 -centralizer is not a topological invariant even among Anosov diffeomorphisms. Indeed, since C^1 -generic Anosov diffeomorphisms have trivial centralizer then $N^1(f) = 1$ for a C^1 -generic set in $\text{Diff}^1(M)$. On the other hand, there are C^∞ -Anosov diffeomorphisms with a discrete but non-trivial C^1 -centralizer [20].

If f is structurally stable then Theorem 2 implies that the uniqueness of conjugacies C^0 -close to the identity is equivalent to the C^0 -centralizer to be discrete. This, together with Theorem 2.1, yields the following consequence.

Corollary 3. *Let $f \in \text{Diff}^1(M)$ be a structurally stable diffeomorphism. If $Z^0(f)$ is discrete then:*

- (a) *there exists $\varepsilon > 0$ and an open neighborhood $\mathcal{U} \subset \text{Diff}^1(M)$ so that for any $g \in \mathcal{U}$ there exists a unique conjugacy $h_g \in \mathcal{H}_{f,g}$ (that is, $h_g \circ f = g \circ h_g$) which is ε - C^0 -close to the identity;*
- (b) *the function $\mathcal{U} \ni g \mapsto h_g \in \text{Homeo}(M)$ is continuous;*
- (c) *there exists $K > 0$ so that $d_{C^0}(h_g, \text{id}) \leq K d_{C^0}(g, f)$; and*
- (d) *for any $g \in \mathcal{U}$ the map $\Psi_{f,g}: Z^0(f) \rightarrow Z^0(g)$ given by $\Psi_{f,g}(\tilde{f}) = h_g^{-1} \circ \tilde{f} \circ h_g$ is an homeomorphism.*

The continuous dependence of the conjugacy map h_g on the diffeomorphism g follows from the work of Robbin and Robinson [22, 23].

In what follows we relate expansiveness to C^0 -centralizers for structurally stable diffeomorphisms. Our starting point is the following result due to Walters [28]:

Lemma 3.1. *Take $r \geq 0$. Assume $f \in \text{Diff}^r(M)$ has an f -invariant subset Λ for which $f|_\Lambda$ is expansive. For every $0 \leq k \leq r$, the subgroup $Z^k(f|_\Lambda)$ of $\text{Homeo}(\Lambda)$ is discrete. In particular, $Z^0(f|_{\Omega(f)})$ is discrete for every C^r Axiom A diffeomorphism and $1 \leq r \leq \infty$.*

We observe that taking f restricted to the non-wandering set $\Omega(f)$ above is necessary (see Example 7.1) and that one cannot expect triviality of C^0 -centralizers even for Anosov diffeomorphisms [24].

3.1. Densely expansive structurally stable diffeomorphisms. We get back to expansiveness and structural stability. Mañé [15] proved that an expansive structurally stable diffeomorphism is Anosov. In this subsection we prove this is still the case for structurally stable diffeomorphisms that admit an *invariant* and dense set of expansiveness. We start with the following characterization of structurally stable diffeomorphisms with discrete C^0 -centralizer.

Theorem 3. *If $f \in \text{Diff}^1(M)$ is a structurally stable diffeomorphism then the following are equivalent:*

- (i) $Z^0(f)$ is discrete;
- (ii) for any $g \in C_f$ there exists a unique conjugacy $h_g \in \text{Homeo}(M)$ that is C^0 -close to the identity;
- (iii) if $\Lambda_{i_s} > \dots > \Lambda_{i_2} > \Lambda_{i_1}$ is a maximal ordered chain of basic pieces in $\Omega(f)$ then $\dim W^u(\Lambda_{i_j}) = \dim W^u(\Lambda_{i_1})$ is constant for every $1 \leq j \leq s$; and
- (iv) f is an Anosov diffeomorphism.

Actually, the later result can be used to deduce that Mañé's result in [15] is optimal: there are structurally stable diffeomorphisms with a dense subset of points of expansiveness which are not Anosov (cf. Example 7.2).

Theorems 1 and 3 together yield the following immediate consequence:

Corollary 4. *Assume $f \in \text{Diff}^1(M)$ is a structurally stable diffeomorphism so that the basins of trivial attractors and trivial repellers do not intersect and that there exists a maximal totally ordered chain $\Lambda_{i_s} > \dots > \Lambda_{i_2} > \Lambda_{i_1}$ of basic pieces in $\Omega(f)$ satisfying $\dim W^u(\Lambda_{i_s}) > \dim W^u(\Lambda_{i_1})$. Then*

- (a) $Z^0(f)$ is not discrete,
- (b) f has sensitivity to initial conditions,
- (c) f admits no f -invariant and dense set of expansiveness.

The topological description of the foliations and attractors for Axiom A diffeomorphisms is a hard topic not yet completely understood. Nevertheless, structurally stable diffeomorphisms that satisfy the assumptions of Corollary 4 include: (i) structurally stable diffeomorphisms of a compact orientable surface that contains a one-dimensional basic set (because these always admit also periodic sources or sinks (cf. [19, 8])), and (ii) structurally stable diffeomorphisms on compact manifolds of dimension larger or equal to 3 with a codimension one orientable expanding attractor (since these have at least one periodic repeller and all remaining basic pieces in the spectral decomposition are periodic points as proved in [9]).

4. DENSE EXPANSIVENESS FOR STRUCTURALLY STABLE DIFFEOMORPHISMS

This section is devoted to the proof of Theorem 1. Let f be a structurally stable diffeomorphism. We first recall the dynamics of f on wandering points.

4.1. Points traveling through filtration elements. Since f is Axiom A, by the spectral decomposition, the non-wandering set is the union of transitive hyperbolic basic pieces: $\Omega(f) = \bigcup_{i=1}^k \Lambda_i$. We denote by $\mathcal{A}(f) \subset \Omega(f)$ be the set of attractors for f . The strong transversality condition implies that $W^s(x)$ is transversal to $W^u(y)$ for every $x, y \in \Omega(f)$ which implies, if these have non-empty intersection, that $\dim E_x^s + \dim E_y^u \geq \dim M$. By the no cycles condition, hence existence of a filtration adapted to $\Omega(f)$, any maximal totally ordered chain $\Lambda_{i_s} > \dots > \Lambda_{i_2} > \Lambda_{i_1}$ has at most k elements, the basic set Λ_{i_s} is a repeller, the set Λ_{i_1} is an attractor, and the remaining elements Λ_{i_j} are of saddle type (see e.g. [25]).

Remark 4.1. *For any chain $\Lambda_{i_s} > \dots > \Lambda_{i_2} > \Lambda_{i_1}$ there are points $x_{i_j} \in \Lambda_{i_j}$ so that $W^u(x_{i_{j+1}}) \cap W^s(x_{i_j}) \neq \emptyset$ for all $j = 1 \dots s-1$. Since periodic points are dense in $\Omega(f)$, invariant manifolds vary continuously on compact parts and transversality in compact parts is an open condition, we may assume the points x_{i_j} to be periodic.*

The following instrumental lemma allow us to describe the set of points that accumulate, by backward orbits, on the basic sets of saddle type. First we need to introduce some notation. Given an attractor Λ , let $\mathfrak{C}(\Lambda)$ denote the set of all maximal and totally ordered chains $\Lambda_{i_s} > \dots > \Lambda_{i_2} > \Lambda_{i_1}$ of basic pieces in the non-wandering set $\Omega(f)$ such that $\Lambda_{i_1} = \Lambda$. Observe that the set Λ_{i_s} is a repeller.

Lemma 4.1. *Let $f \in \text{Diff}^1(M)$ be a structurally stable diffeomorphism and let $\Lambda \subset \Omega(f)$ be an attractor for f . Then for any periodic point $p \in \Lambda$,*

$$W^s(p) \setminus \left[\bigcup_{\Lambda_{i_s} > \dots > \Lambda_{i_2} > \Lambda_{i_1} \in \mathfrak{C}(\Lambda)} \bigcup_{2 \leq j \leq s-1} W^u(\Lambda_{i_j}) \right] \quad (4.1)$$

forms an f -invariant open and dense subset of $W^s(p)$. In particular, there exists an open and dense subset of points in $W^s(p)$ whose backward orbit accumulates in some of the repellers heteroclinically related to Λ .

Proof. Since $f \in \text{Diff}^1(M)$ is structurally stable, the basic pieces in $\Omega(f)$ are locally maximal and

$$M = \bigcup_{x \in \Omega(f)} W^s(x) = \bigcup_{x \in \Omega(f)} W^u(x). \quad (4.2)$$

Let $\Lambda \subset \Omega(f)$ be an attractor, p be a periodic point in Λ and let D be the set defined by (4.1). We claim that D is dense in $W^s(p)$. Given $x \in W^s(p)$, by (4.2) and the strong transversality condition, there exists $p_x \in \Omega(f)$ so that $x \in W^u(p_x) \cap W^s(p)$ (see Figure 1 below).

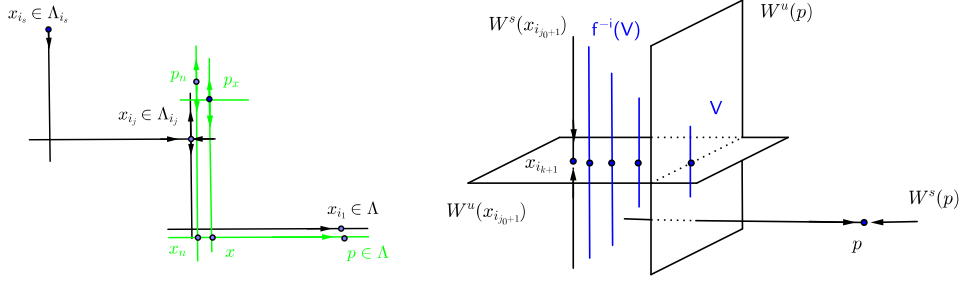


FIGURE 1. Heteroclinic intersections associated to the basic pieces (on the left) and selection of a disk $V \subset W^u(x_{i_k})$ accumulating on $W^s(x_{i_{j_0+1}})$ by the λ -lemma (on the right).

Let $\Lambda_{i_s} > \dots > \Lambda_{i_2} > \Lambda_{i_1} \in \mathfrak{C}(\Lambda)$, $2 \leq j \leq s$ be a maximal chain containing the basic set $\Lambda_{i_{j_0}}$ that contains p_x . If $j_0 = s$ then $d(f^{-k}(x), \Lambda_{i_s}) \rightarrow 0$ as $k \rightarrow \infty$, and so $x \in D$ and we are done. Otherwise $2 \leq j_0 < s$ and, by the denseness of periodic points in the non-wandering set and continuity of compact parts of stable and unstable manifolds, one can take periodic points $p_n \in \Lambda_{i_{j_0}}$ so that $p_n \rightarrow p_x$, and x is approximated by heteroclinic points $x_n \in W^u(p_n) \cap W^s(p)$. By Remark 4.1 there are periodic points $x_{i_k} \in \Lambda_{i_k}$ so that $W^u(x_{i_{k+1}}) \cap W^s(x_{i_k})$ for all $k = j_0 \dots s-1$. By the λ -lemma (see e.g. [25]), there exists a disk $V \subset W^s(p)$ of dimension equal to the stable index of $\Lambda_{i_{j_0+1}}$ whose iterates by f^{-1} accumulate (in the C^1 -topology) on a compact part of $W^s(x_{i_{j_0+1}})$ (see Figure 1). A recursive argument assures that x is accumulated by heteroclinic intersections between the stable manifold of p and unstable manifolds of periodic points in Λ_{i_s} . Since these homoclinic intersections belong to D , this proves that D is dense in $W^s(p)$.

We are left to prove that D is open in $W^s(p)$. Given $x \in D$ arbitrary, let $n_0 \geq 1$ be so that $f^{-n_0}(x)$ belongs to the immediate topological basin of the repeller. Since the latter is an open set, by continuity of f^{-1} , there exists an open neighborhood V of x so that $f^{-n_0}(V)$ is also contained in the topological basin of the repeller. This implies that D is an open subset of $W^s(p)$ and finishes the proof of the lemma. \square

Remark 4.2. *This result asserts that ‘most’ wandering points travel along the elements of the filtration and converge by negative iteration to some of the repellers. Clearly, a similar statement holds for the repellers of f just by considering the previous lemma for the diffeomorphism f^{-1} .*

4.2. Proof of Theorem 1. Let $f \in \text{Diff}^1(M)$ be a C^1 -structurally stable diffeomorphism.

First part: (1) \Rightarrow (2). Assume that the topological basins of trivial attractors and repellers in $\Omega(f)$ do not intersect. Let $\Omega(f) = \overline{\text{Per}(f)} = \bigcup_{i=1}^k \Lambda_i$ be the spectral decomposition for $\Omega(f)$, where each Λ_i is the homoclinic class associated to some periodic point $p_i \in \text{Per}(f) \cap \Lambda_i$. Since all periodic points are homoclinically related

then: (i) $\Lambda_i = H(p_i)$ (hence $W^s(p_i)$ is dense in Λ_i) for every $i = 1 \dots k$; (ii) $W^s(\Lambda_i) := \{x \in M : d(f^n(x), \Lambda_i) \rightarrow 0 \text{ as } n \rightarrow \infty\}$ is an open subset of M for every $\Lambda_i \in \mathcal{A}(f)$, and (iii) $W^s(p_i)$ is dense in $W^s(\Lambda_i)$. Moreover, if $\varepsilon_i > 0$ is given by the expansiveness property for $f|_{\Lambda_i}$ then we conclude that $f|_{\Omega(f)}$ is ε -expansive, for $\varepsilon = \min\{\varepsilon_i : 1 \leq i \leq k\} > 0$. By Lemma 4.1, one can choose a finite set $\Theta = \Theta_a \cup \Theta_r$ of periodic points, Θ_a chosen on the attractors and Θ_r chosen on the repellers of the spectral decomposition of $\Omega(f)$ such that:

- (1) for $p \in \Theta_a$, $\overline{W^s(p)} = W^s(H(p, f))$ and contains an open set in the basin of some non-trivial repeller,
- (2) for $p \in \Theta_r$, $\overline{W^u(p)} = W^u(H(p, f))$ and contains an open set in the basin of some non-trivial attractor,
- (3) the union of topological basins of the non-trivial repellers contains an open and dense subset of the stable manifolds $\bigcup_{p \in \Theta_a} W^s(p)$,
- (4) the union of the topological basin of non-trivial attractors contains an open and dense subset of the unstable manifolds $\bigcup_{p \in \Theta_r} W^u(p)$,
- (5) $[\bigcup_{p \in \Theta_a} W^s(p)] \cup [\bigcup_{p \in \Theta_r} W^u(p)]$ is dense in M .

By construction, the union of topological basins of the non-trivial repellers contains an open and dense subset $O_p \subset W^s(p)$ for any $p \in \Theta_a$ (and a similar statement holds for $p \in \Theta_r$). Consider the set P given as the union of all periodic points in the non-trivial attractors of f that are heteroclinically related to some periodic point in Θ_r and all periodic points in the non-trivial repellers of f that are heteroclinically related to some periodic point in Θ_r . There are countable infinitely many of such periodic points. Now, since any countable infinite set is a disjoint countable union of countable infinite sets and $\sharp(\Theta_a \cup \Theta_r) < \infty$ one can decompose

$$P = \bigsqcup_{p \in \Theta_a \cup \Theta_r} \bigsqcup_{\ell \in \mathbb{Z}} P_{p,\ell} \quad (4.3)$$

as a disjoint union of countable infinite sets $P_{p,\ell} \subset P$ in such a way that for each $p \in \Theta_a$ (resp. for $p \in \Theta_r$) the set $P_{p,\ell}$ contains infinitely many periodic points in each of the non-trivial repellers whose basin intersects $W^s(p)$ (resp. non-trivial attractors whose basin intersects $W^u(p)$). We use the following key claim:

Claim: For every periodic point $p \in \Theta_a$ there exists a countable and dense subset $D_p \subset W^s(p)$ so that the following holds: for any $x, y \in D_p$ there are periodic points $p_x \neq p_y$ in $\bigsqcup_{\ell \in \mathbb{Z}} P_{p,\ell}$ such that $x \in W^u(p_x)$ and $y \in W^u(p_y)$. In particular, there exists $n = n_{x,y} \geq 1$ for which $d(f^{-n}(x), f^{-n}(y)) > \frac{\varepsilon}{2}$.

Since claim also holds for the periodic points in Θ_r (replacing f by f^{-1} , in which case unstable manifolds for f become stable manifolds for f^{-1}), by the decomposition of periodic points in (4.3) we conclude that there exists a countable and dense subset of $\bigcup_{p \in \Theta_a} W^s(p) \cup \bigcup_{p \in \Theta_r} W^u(p)$ formed by points that are $\frac{\varepsilon}{2}$ separated by either positive or negative iterations of f . By item (5) above, the later set is dense in M . Thus, in order to complete the proof of the first part of the theorem we are left to prove the claim.

Proof of the Claim. Fix $p \in \Theta_a$ and let $\Delta \subset W^s(p)$ be a fundamental domain of $W^s(p)$. Given $\ell \in \mathbb{Z}$ define $\Delta_\ell = f^\ell(\Delta)$. Observe that $W^s(p) \setminus \{p\} = \bigsqcup_{\ell \in \mathbb{Z}} \Delta_\ell$. Fix $\ell \in \mathbb{Z}$ and set $P_\ell^1 := P_{p,\ell}$. Given any open covering of $\overline{\Delta_\ell}$ by open balls of radius $1/2$, the compactness of the closure $\overline{\Delta_\ell}$ guarantees one can extract a finite covering $\{B_i^2\}_i$. The λ -lemma (as used in the proof of Lemma 4.1) implies all points in $\overline{\Delta_\ell}$ are accumulated by heteroclinic intersections between the unstable manifolds of every periodic point in P_ℓ^1 and the stable manifold of p . Thus for every i there exists $p_i^2 \in P_\ell^1$ so that the intersection $W^u(p_i^2) \cap W^s(p)$ contains some point $x_{i,2}$ in B_i^2 . Since all periodic points in the same homoclinic class are related, we choose the periodic points $\{p_{i,2}\}_i$ to be distinct. Moreover, by construction, the set $\{x_{i,2}\}_i$ (we omit the dependence on ℓ for notational simplicity) is finite and $1/2$ -dense in $\overline{\Delta_\ell}$. Moreover the set $P_\ell^2 := P_\ell^1 \cup \{p_i^2\}$ has still infinitely many periodic points in each of the repellers heteroclinically related to the attractor Λ that contains p .

Proceeding recursively, for every $m \geq 1$ we obtain a finite number of periodic points $\{p_i^m\} \in P_\ell^m$ whose heteroclinic intersections of the corresponding unstable manifolds with $W^s(p)$ contains a $1/m$ -dense set $\{x_{i,m}\}_i$ of points in $\overline{\Delta_\ell}$, and the set $P_\ell^{m+1} := P_\ell^m \cup \{p_i^m\}$ has still infinitely many periodic points in each of the repellers heteroclinically related to Λ . The resulting set $D_{p,\ell} := \bigcup_{m \geq 1} \{x_{i,m}\}$, formed by points obtained by the heteroclinic intersections, is a countable dense subset in $\overline{\Delta_\ell}$ and the set $D_p := \bigcup_{\ell \geq 1} D_{p,\ell}$ satisfies the requirements of the

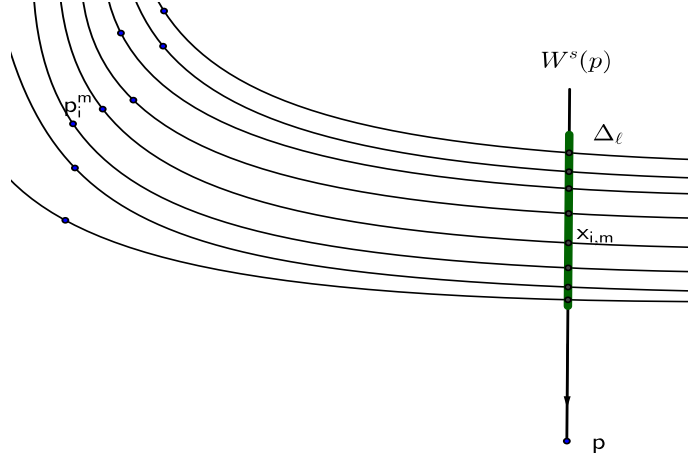


FIGURE 2. Representation of $\frac{1}{m}$ -dense heteroclinic intersections $\{x_{i,m}\}$ associated to periodic points in some non-trivial repeller in the fundamental domain Δ_ℓ of $W^s(p)$.

lemma: for any $x, y \in D_p$ there are periodic points $p_x \neq p_y \in \text{Per}(f)$ such that $x \in W^u(p_x)$ and $y \in W^u(p_y)$. In consequence,

$$d(f^{-n}(x), f^{-n}(y)) \geq d(f^{-n}(p_x), f^{-n}(p_y)) - d(f^{-n}(p_x), f^{-n}(x)) - d(f^{-n}(p_y), f^{-n}(y))$$

can be taken larger than $\varepsilon/2$ provided that n is large enough (here we used that the set $\text{Per}(f)$ is negatively expansive by f). This completes the proof of the lemma. \square

Second part: (2) \Rightarrow (3). This is immediate.

Third part: (3) \Rightarrow (1). We prove this by contradiction. Assume that there are periodic points p_1, p_2 so that $\Lambda_1 = O(p_1)$ is a trivial repeller, $\Lambda_2 = O(p_2)$ is a trivial attractor and $W^u(\Lambda_1) \cap W^s(\Lambda_2) \neq \emptyset$. By uniform continuity, f is sensitive to initial conditions if and only if f^k ($k \geq 1$) is sensitive to initial conditions (although the constants of separability could change). Hence, replacing f by some suitable iterate f^k , we may assume without loss of generality that both p_1, p_2 are fixed points for f . The existence of a filtration guarantees that $W^u(\Lambda_1) \cap W^s(\Lambda_2)$ contains a non-empty open set V . We claim that for any $\varepsilon > 0$ there exists an open subset $V_\varepsilon \subset V$ of points so that $d(f^n(x), f^n(y)) < \varepsilon$ for all $n \in \mathbb{Z}$. Indeed, if $k = k(\varepsilon, V) \geq 1$ is so that $f^j(V) \subset B(p_2, \varepsilon)$ and $f^{-j}(V) \subset B(p_1, \varepsilon)$ for every $j \geq k$, and $x \in V$ it is enough to take the open set

$$V_\varepsilon = \{y \in V : d(f^j(x), f^j(y)) < \varepsilon, \forall |j| \leq k\}.$$

This proves that there exists an open set of points whose iterates always remain ε -close. Since ε was chosen arbitrary this contradicts the sensitivity to initial conditions assumption on f . The proof of the theorem is now complete.

5. C^0 -CONJUGACY CLASSES AND C^0 -CENTRALIZERS OF HOMEOMORPHISMS

5.1. Proof of Theorem 2. Fix $r \geq 0$ and $f \in \text{Diff}^r(M)$. For any $g \in C_f$ and $h \in \mathcal{H}_{f,g}$, consider the map

$$F_h: \begin{array}{ccc} Z^0(f) & \rightarrow & \mathcal{H}_{f,g} \\ \tilde{f} & \mapsto & h \circ \tilde{f}, \end{array}$$

where the subsets $Z^0(f), \mathcal{H}_{f,g}$ of $\text{Homeo}(M)$ are endowed with the distance d_0 . We claim that F_h is well defined. Indeed, for any $\tilde{f} \in Z^0(f)$ it holds $\tilde{f} \circ f = f \circ \tilde{f}$. Thus $(h \circ \tilde{f}) \circ f = h \circ (\tilde{f} \circ f) = (h \circ f) \circ \tilde{f} = (g \circ h) \circ \tilde{f} = g \circ (h \circ \tilde{f})$. This can be also observed by the commutative diagram;

$$\begin{array}{ccccc}
M & \xrightarrow{\tilde{f}} & M & \xrightarrow{h} & M \\
f \downarrow & & f \downarrow & & g \downarrow \\
M & \xrightarrow{\tilde{f}} & M & \xrightarrow{h} & M
\end{array}$$

This proves that $h \circ \tilde{f}$ is also a C^0 -conjugacy between f and g . We proceed to prove that F_h is indeed an homeomorphism. Since h is an homeomorphism then F_h is clearly injective. To prove the continuity of F_h , fix an arbitrary $\tilde{f} \in Z^0(f)$. By the uniform continuity of h , given $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ so that $d(h(x), h(y)) < \varepsilon$ for any points $x, y \in M$ satisfying $d(x, y) < \delta$. In particular, if $\tilde{f}, \hat{f} \in Z^0(f)$ and $d_0(\tilde{f}, \hat{f}) < \delta$ then

$$\begin{aligned}
d_0(F_h(\tilde{f}), F_h(\hat{f})) &= \max\{\sup_{x \in M} d(h(\tilde{f}(x)), h(\hat{f}(x))), \sup_{x \in M} d(\tilde{f}^{-1}(h^{-1}(x)), \hat{f}^{-1}(h^{-1}(x)))\} \\
&< \max\{\varepsilon, d_0(\tilde{f}, \hat{f})\} = \varepsilon,
\end{aligned}$$

which proves the continuity of F_h . To prove that F_h is surjective, given $\tilde{h} \in \mathcal{H}_{f,g}$ write $\tilde{h} = h \circ (h^{-1} \circ \tilde{h})$ and note that $h^{-1} \circ \tilde{h}$ is an homeomorphism. Moreover

$$f \circ (h^{-1} \circ \tilde{h}) = (f \circ h^{-1}) \circ \tilde{h} = (h^{-1} \circ g) \circ \tilde{h} = h^{-1} \circ (g \circ \tilde{h}) = (h^{-1} \circ \tilde{h}) \circ f,$$

which can also be read from the commutative diagram

$$\begin{array}{ccccc}
M & \xrightarrow{\tilde{h}} & M & \xrightarrow{h^{-1}} & M \\
f \downarrow & & g \downarrow & & f \downarrow \\
M & \xrightarrow{\tilde{h}} & M & \xrightarrow{h^{-1}} & M
\end{array}$$

This proves that $h^{-1} \circ \tilde{h} \in Z^0(f)$, and so F_h is a continuous bijection whose inverse map is $F_h^{-1} : \mathcal{H}_{f,g} \rightarrow Z^0(f)$ given by $F_h^{-1}(\tilde{h}) = h^{-1} \circ \tilde{h}$. Since h^{-1} is continuous, hence uniformly continuous, then F_h^{-1} is clearly continuous. Altogether this proves that F_h is an homeomorphism and finishes the proof of the first part of the proposition.

A simple computation shows that $F_h(\tilde{f} \circ \hat{f}) = h \circ (\tilde{f} \circ \hat{f}) = (h \circ \tilde{f}) \circ (h \circ \hat{f}) = F_h(\tilde{f}) \circ h^{-1} \circ F_h(\hat{f})$ for every $\tilde{f}, \hat{f} \in Z^0(f)$. Property (i) is immediate. Property (iii) is a direct consequence of the fact that homeomorphisms preserve discrete sets. We are left to prove property (ii). Since the elements in the decomposition $\mathcal{H}_f = \bigcup_{g \in C_f} \mathcal{H}_{f,g}$ are pairwise disjoint, for any $h \in \mathcal{H}_f$ there exists a unique $g \in C_f$ so that $h \in \mathcal{H}_{f,g}$ (we write for simplicity $h = h_g$). The map

$$\begin{aligned}
F: \quad \mathcal{H}_f &\rightarrow C_f \\
h = h_g &\mapsto g
\end{aligned}$$

is continuous. Indeed, given $\varepsilon > 0$ take $\delta = \varepsilon/2$ and assume that $d_0(h, \tilde{h}) < \delta$, where $h = h_g$ and $\tilde{h} = \tilde{h}_{\tilde{g}}$. Observe that $g \circ (h_g \circ \tilde{h}_{\tilde{g}}^{-1}) = (h_g \circ \tilde{h}_{\tilde{g}}^{-1}) \circ \tilde{g}$ (in other words $\tilde{h} := h_g \circ \tilde{h}_{\tilde{g}}^{-1} \in \mathcal{H}_{\tilde{g},g}$). Moreover, $d(\tilde{h}(x), x) = d(h_g(\tilde{h}_{\tilde{g}}^{-1}x), \tilde{h}_{\tilde{g}}(\tilde{h}_{\tilde{g}}^{-1}x)) < \delta$ for every $x \in M$. This, together with identical computations for \tilde{h}^{-1} implies that $d_0(\tilde{h}, id) < \delta$. Thus,

$$d_0(g, \tilde{g}) = d_0(\tilde{h} \circ \tilde{g} \circ \tilde{h}^{-1}, \tilde{g}) \leq d_0(\tilde{h} \circ \tilde{g} \circ \tilde{h}^{-1}, \tilde{g} \circ \tilde{h}^{-1}) + d_0(\tilde{g} \circ \tilde{h}^{-1}, \tilde{g}) < \varepsilon.$$

This proves the continuity of F . Since $F^{-1}(\{g\}) = \mathcal{H}_{f,g} \simeq Z^0(f)$ for every $g \in C_f$ then we conclude that \mathcal{H}_f is homeomorphic to $C_f \times Z^0(f)$. This proves (ii) and completes the proof of the proposition. \square

Although the argument used in the proof of Lemma 3.1 is essentially contained in [28], we include it here for completeness.

5.2. Proof of Lemma 3.1. Take $r \geq 0$ and assume $f \in \text{Diff}^r(M)$ has an f -invariant subset Λ such that $f|_\Lambda$ is expansive. Fix $0 \leq k \leq r$. By the inclusion $Z^0(f|_\Lambda) \supset Z^k(f|_\Lambda)$ it is enough to prove that $Z^0(f|_\Lambda)$ is a discrete subgroup of $\text{Homeo}(\Lambda)$. Due to the subgroup structure in $\text{Homeo}(M)$, in order to prove that $Z^0(f|_\Lambda)$ is discrete it is enough to prove that there exists $\delta > 0$ so that any $h \in Z^0(f|_\Lambda)$ with $d_0(h, id_\Lambda) < \delta$ coincides with the identity map id_Λ . Let $\varepsilon > 0$ be an expansiveness constant for $f|_\Lambda$ and $\delta = \varepsilon/2$. Take $h \in Z(f|_\Lambda)$ so that $d_0(h, id|_\Lambda) < \delta$. Since Λ is an f -invariant set and $h \circ f = f \circ h$ on Λ then

$$d(f^n(h(x)), f^n(x)) = d(h(f^n(x)), f^n(x)) \leq d_0(h, id) < \varepsilon/2$$

for any $x \in \Lambda$ and every $n \in \mathbb{Z}$. By expansiveness it follows that $h = id|_\Lambda$. To finish the proof of the lemma, just observe that f is an Axiom A diffeomorphism then $\Omega(f)$ consists of a finite number of hyperbolic, hence expansive, homoclinic classes. In particular, $Z^0(f|_{\Omega(f)})$ is discrete for every C^r Axiom A diffeomorphism and $1 \leq r \leq \infty$. This completes the proof of the lemma.

Remark 5.1. In the case of an Axiom A diffeomorphism f there exists $\varepsilon > 0$ and a C^1 neighborhood \mathcal{U} of f so that the expansiveness constant ε_g for $g|_{\Omega(g)}$ is uniformly bounded below by $\varepsilon > 0$ for all $g \in \mathcal{U}$. Thus, there exists $\varepsilon > 0$ so that $d_0(h, id) > \varepsilon$ for any $h \in Z^0(g|_{H(p_{g,g})})$ and $g \in \mathcal{U}$. This means that the smallest distance to identity in $Z^0(f|_{\Omega(f)}) \setminus \{id\}$ can be taken uniform in a small neighborhood of f .

6. DENSE EXPANSIVENESS AND CONJUGACIES C^0 -CLOSE TO IDENTITY

The purpose of this section is to prove Theorem 3 and their consequences.

6.1. Proof of Theorem 3. Let $f \in \text{Diff}^1(M)$ be a structurally stable diffeomorphism, hence Axiom A. It follows from the spectral decomposition theorem for Axiom A diffeomorphisms that $\Omega(f) = \bigcup_{i=1}^k \Lambda_i$ where each Λ_i is a hyperbolic homoclinic class associated to a periodic point $p_i \in \text{Per}(f)$. The equivalence between (i) and (ii) follows from Theorem 2. Here we prove the remaining equivalences as follows:

(iii) \Rightarrow (iv)

Assume that for any maximal totally ordered chain $\Lambda_{i_s} > \dots > \Lambda_{i_2} > \Lambda_{i_1}$ of basic pieces in $\Omega(f)$ one has $\dim W^u(\Lambda_{i_j}) = \dim W^u(\Lambda_{i_1})$ is constant for every $1 \leq j \leq s$. Since f is Axiom A and satisfies the strong transversality condition, for any $x \in M$ there exist $y, z \in \Omega(f)$ uniquely determined so that $x \in W^u(y) \pitchfork W^s(z)$. In particular, $\dim W^u(y) + \dim W^s(z) = \dim M$ and there exists $\varepsilon_x > 0$ so that $\mathcal{F}_{\varepsilon_x}^u(x) \pitchfork \mathcal{F}_{\varepsilon_x}^s(x) = \{x\}$, where $\mathcal{F}_{\varepsilon_x}^u(x)$ denotes the ε_x -ball around x in $W^u(y)$ and $\mathcal{F}_{\varepsilon_x}^s(x)$ denotes the ε_x -ball around x in $W^s(z)$. Since M is compact and the invariant foliations vary continuously with the point on compact parts then there exists $\varepsilon_0 > 0$ so that $\mathcal{F}_{\varepsilon_0}^u(x) \pitchfork \mathcal{F}_{\varepsilon_0}^s(x) = \{x\}$ for every $x \in M$. The later implies that ε_0 is an expansiveness constant for f . By [15], f is an Anosov diffeomorphism.

(iv) \Rightarrow (i)

Since every Anosov diffeomorphism is expansive, (i) follows from Lemma 3.1.

(i) \Rightarrow (iii)

If f is an Anosov diffeomorphism then both properties (i) and (iii) hold. For that reason we need only to consider C^1 -structurally stable diffeomorphisms that are not Anosov. By structural stability, for any a maximal totally ordered chain $\tilde{\Lambda}_{i_s} > \dots > \tilde{\Lambda}_{i_2} > \tilde{\Lambda}_{i_1}$ of basic pieces in $\Omega(f)$ recall that $\dim W^u(\tilde{\Lambda}_{i_j}) \leq \dim W^u(\tilde{\Lambda}_{i_{j+1}})$ for every $1 \leq j \leq s-1$. We prove this implication by contraposition. Assume that there exists a maximal totally ordered chain $\Lambda_{i_s} > \dots > \Lambda_{i_2} > \Lambda_{i_1}$ of basic pieces in $\Omega(f)$ so that $\dim W^u(\Lambda_{i_1}) < \dim W^u(\Lambda_{i_s})$. Our purpose is to prove that $Z^0(f)$ is not discrete.

Since $\Omega(f)$ is a hyperbolic set for f (hence $f|_{\Omega(f)}$ is expansive) there exists $\varepsilon_0 > 0$ an expansiveness constant for $f|_{\Omega(f)}$. Given any element $h \in Z^0(f)$ it preserves the non-wandering set of f , that is $h(\Omega(f)) = \Omega(f)$. Therefore, if in addition it satisfies $d_{C^0}(h, id) < \varepsilon_0$ then $h|_{\Omega(f)} = id$ (cf. Lemma 3.1). Furthermore, since any element in $Z^0(f)$ preserves stable and unstable foliations then $h(W^s(x)) = W^s(x)$ and $h(W^u(x)) = W^u(x)$ for every $x \in \Omega(f)$. In particular h preserves heteroclinic points.

The remaining of the proof is constructive and we build a continuum of homeomorphisms C^0 -close to the identity that belong to $Z^0(f)$. By Lemma 4.1, there exists a periodic point in Λ_{i_1} that is heteroclinically related to some periodic point in Λ_{i_s} . Moreover, since compact parts of stable and unstable manifolds vary continuously with the point and there exists a filtration, the argument used in the final part of the proof of Lemma 4.1 guarantees that there exists an open set $U \subset M \setminus \Omega(f)$ so that $\text{dist}_H(f^n(U), \Lambda_{i_1}) \rightarrow 0$ and $\text{dist}_H(f^{-n}(U), \Lambda_{i_s}) \rightarrow 0$ as n tends to infinity (here dist_H denotes the Hausdorff distance). Since the sets Λ_{i_1} and Λ_{i_s} are locally maximal then $W^s(\Lambda_{i_1}) = \bigcup_{x \in \Lambda_{i_1}} W^s(x)$ and $W^u(\Lambda_{i_s}) = \bigcup_{x \in \Lambda_{i_s}} W^u(x)$. In particular every point in U belongs to a stable manifold of some point of Λ_{i_1} and to an unstable manifold of some point of Λ_{i_s} . Diminishing U if necessary, we may assume that the collection of compact sets $(\overline{f^n(U)})_{n \in \mathbb{N}}$ is pairwise disjoint, and that U is foliated by pieces of stable and unstable disks.

Since we assumed $\dim W^u(\Lambda_{i_1}) < \dim W^u(\Lambda_{i_s})$ then $\dim W^u(\Lambda_{i_s}) + \dim W^s(\Lambda_{i_1}) > \dim M$ for every $x \in U$. Let $\alpha_x \subset U$ be a smooth submanifold of dimension one contained in the intersection $\mathcal{F}^u(x) \cap \mathcal{F}^s(x)$ (see Figure 3 below). Up to reduce U if necessary, by continuity of the intersections between invariant manifolds one may

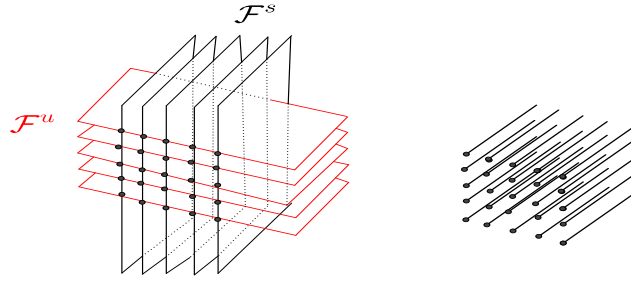


FIGURE 3. Representation of the intersection of pieces of stable and unstable foliations in U (on the left), and smooth submanifolds contained in their intersection (on the right).

choose the family $(\alpha_x)_{x \in U}$ to vary continuously with x and to partitionate U . By convention, $\alpha_x = \alpha_z$ for every $z \in \alpha_x$. Up to conjugacy by a homeomorphism on U , there exist natural coordinate system given by arc length. More precisely, if $\ell(\alpha_x)$ denotes the length of the one-dimensional submanifold α_x and $E \subset W^u(\Lambda_{i_s}) \cap U$ is the continuous submanifold that intersects each submanifold α_x exactly on its midpoint, the parametrization of α_x by arc length with the same orientation for all points (which we denote by $\alpha_x :] - \ell(\alpha_x)/2, \ell(\alpha_x)/2[\rightarrow M$ by some abuse of notation) induces a natural homeomorphism

$$\begin{aligned} \varphi : U &\rightarrow \tilde{U} := \{(x, \ell) \in E \times \mathbb{R} : x \in E, -\ell(\alpha_x)/2 < \ell < \ell(\alpha_x)/2\} \\ z &\mapsto (x, \ell) \end{aligned}$$

where $\alpha_z \cap E = \{x\}$ and $-\ell(\alpha_x)/2 < \ell < \ell(\alpha_x)/2$ is unique so that $\alpha_x(\ell) = z$. Now, let $V \subsetneq W \subsetneq U$ be two small open sets so that $V \subsetneq \overline{V} \subsetneq W \subsetneq \overline{W} \subsetneq U$ and $\ell(\alpha_x) \geq \ell_0 > 0$ for all points $x \in V$ (see Figure 4). Observe that $\varphi|_{\overline{W}} : \overline{W} \rightarrow \varphi(\overline{W})$ and its inverse are uniformly continuous. Since every element in the centralizer preserves

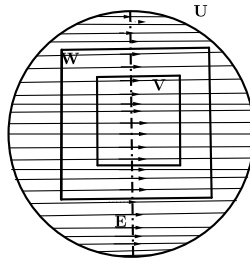


FIGURE 4. Foliated chart

heteroclinic points this motivates a perturbation of the identity map along this one-dimensional fibers. Fix a

continuous ‘bump function’ $\beta : \mathbb{R} \rightarrow [0, 1]$ so that $\beta|_{\mathbb{R} \setminus [-\ell_0/2, \ell_0/2]} \equiv 0$ and $\beta|_{[-\ell_0/4, \ell_0/4]} \equiv 1$. Fix $\zeta > 0$ small and $t \in [0, 1]$. For any $z \in U$, consider the one-parameter family

$$h_{0,t}(z) := \begin{cases} \varphi^{-1} \circ H_t \circ \varphi(z) & , \text{if } z \in W \\ z & , \text{if } z \in U \setminus W \end{cases}$$

where $H_t : W \rightarrow W$ is an given by $(x, \ell) \mapsto (x, \ell + \beta(\ell)t\zeta)$ and $(x, \ell) = \varphi(z)$. In rough terms, the homeomorphism h_t on V pushes t along the direction determined by the oriented submanifolds α_z . For any $z \in M \setminus \bigcup_{n \in \mathbb{Z}} f^n(U)$ define $h_t(z) = z$ and, otherwise, define $h_t(z) = f^{-n}(h_{0,t}(f^n(z)))$ where $n = n(z) \in \mathbb{Z}$ is (unique) so that $f^n(z) \in U$.

By construction $h_0 = Id$ and $(h_t)_{t \in [0,1]}$ is a continuous family such that $f \circ h_t = h_t \circ f$ (equivalently $h_t \in Z^0(f)$) for every $t \in [0, 1]$. In order to complete the proof of the theorem we are left to prove that h_t is an homeomorphism for every $t \in [0, 1]$. Notice that h_t is invertible. We proceed to prove the following:

Claim: h_t is continuous

Proof of the claim: On $M \setminus \overline{\bigcup_{n \in \mathbb{Z}} f^n(W)}$ we have that $h_t = id$ is clearly continuous. Since h_t is also continuous on each open set of the form $f^n(W)$, $n \in \mathbb{Z}$, it remains to prove the continuity on points that belong to either the attractor Λ_{i_1} or the repeller Λ_{i_s} . Let $\lambda \in (0, 1)$ denote a hyperbolicity constant for $\Omega(f)$.

Fix $x \in \Lambda_{i_1}$ (if $x \in \Lambda_{i_s}$ the computations are analogous, replacing f by f^{-1}) and $\varepsilon > 0$. Pick $0 < \delta < \varepsilon/2$ small so that for any point $y \in \bigcup_{n \in \mathbb{Z}} f^n(W)$ with $d(x, y) < \delta$, the unique $n_y \in \mathbb{Z}$ (and necessarily negative) so that $f^{n_y}(y) \in U$ satisfies $\lambda^{-n_y}\zeta < \varepsilon/2$. This holds because the number of iterates necessary for a point to enter the set W grows to infinity for points sufficiently close to the attractor Λ_{i_1} . With this choice, if $d(x, y) < \delta$ and $y \notin \bigcup_{n \in \mathbb{Z}} f^n(W)$ then $d(h(x), h(y)) = d(x, y) < \varepsilon/2$. Now assume, otherwise, that $d(x, y) < \delta$ and $y \in \bigcup_{n \in \mathbb{Z}} f^n(W)$. By construction, $h_{0,t}(f^{n_y}(y)) \in W^s(f^{n_y}(y))$ and $d(h_{0,t}(f^{n_y}(y)), f^{n_y}(y)) < \zeta$. Therefore, by triangular inequality

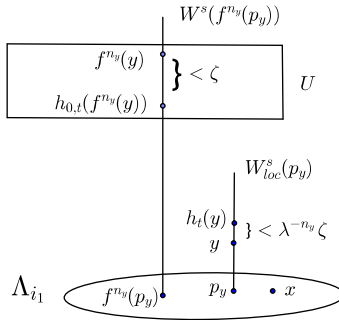


FIGURE 5. Continuity argument

together with the uniform contraction along stable leaves,

$$\begin{aligned} d(h_t(x), h_t(y)) &= d(x, f^{-n_y} \circ h_{0,t} \circ f^{n_y}(y)) \\ &= d(f^{-n_y} \circ f^{n_y}(x), f^{-n_y} \circ h_{0,t} \circ f^{n_y}(y)) \\ &\leq d(f^{-n_y} \circ f^{n_y}(x), f^{-n_y} \circ f^{n_y}(y)) + d(f^{-n_y} \circ f^{n_y}(y), f^{-n_y} \circ h_{0,t} \circ f^{n_y}(y)) \\ &\leq d(x, y) + \lambda^{-n_y} d(f^{n_y}(y), h_{0,t} \circ f^{n_y}(y)) \\ &\leq \delta + \lambda^{-n_y} \zeta < \varepsilon. \end{aligned}$$

This proves the continuity of h_t at x , and completes the proof of the claim. \square

Since the continuity of h_t^{-1} is analogous we conclude that h_t is an homeomorphism C^0 -close to the identity and that $h_t \in Z^0(f)$ for every $t \in [0, 1]$. Therefore, $Z^0(f)_{13}$ is not discrete. This finishes the proof of the theorem.

7. SOME EXAMPLES

Our first example is a Morse-Smale diffeomorphism (hence structurally stable), in which case the C^0 -centralizer contains a continuum of homeomorphisms C^0 -close to identity.

Example 7.1. Let $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a circle diffeomorphism so that the non-wandering set is $\Omega(f) = \{N, S\}$, where N denotes the repelling fixed point and S denotes the contracting fixed point. f is clearly structurally stable. We claim that $Z^0(f)$ contains a continuum C^0 -close to the identity. Since $f|_{\mathbb{S}^1 \setminus \{N\}}$ is homeomorphic to a linear contraction on the real line, we may assume without loss of generality that $f(x) = \frac{1}{2}x$ and fix $I = [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$ as a fundamental domain. For every $x \in \mathbb{S}^1 \setminus \{N\}$ let $n = n(x) \in \mathbb{Z}$ be the unique integer so that $f^n(x) \in I$. Using that $n(f(x)) = n(x) - 1$ for every $x \in \mathbb{R}$ it is not hard to check that every continuous increasing map $h_0 : I \rightarrow I$ with $h_0|_{\partial I} = \text{id}$ determines an element in $Z^0(f)$: the homeomorphism $h : \mathbb{S}^1 \setminus \{N, S\} \rightarrow \mathbb{S}^1 \setminus \{N, S\}$ given by $h(x) = f^{-n(x)}(h_0(f^{n(x)}(x)))$ satisfies

$$\begin{aligned} h(f(x)) &= f^{-n(f(x))}(h_0(f^{n(f(x))}(f(x)))) = f^{-n(x)+1}(h_0(f^{n(x)-1}(f(x)))) \\ &= f(f^{-n(x)}(h_0(f^{n(x)}(x)))) = f(h(x)) \end{aligned}$$

for every $\mathbb{S}^1 \setminus \{N, S\}$, and extends continuously to \mathbb{S}^1 by $h(N) = N$ and $h(S) = S$. As a consequence of Theorem 2, for every $g \in C_f$ there exists a continuum of homeomorphisms conjugating f and g and that are C^0 -close to the identity. Moreover, for any $\varepsilon > 0$ there exists a ball B of radius ε so that $\text{diam } f^n(B) < \varepsilon$ for all $n \in \mathbb{Z}$. This implies that the set of points of ε -expansiveness are not dense in \mathbb{S}^1 .

In the next example we exhibit an open set of densely expansive structurally stable diffeomorphisms with non-discrete C^0 -centralizer.

Example 7.2. Let $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be the Morse-Smale diffeomorphism from Example 7.1 and let $g : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be an Anosov diffeomorphism. It is clear that the diffeomorphism $f \times g : \mathbb{S}^1 \times \mathbb{T}^2 \rightarrow \mathbb{S}^1 \times \mathbb{T}^2$ is an Axiom A C^1 -diffeomorphism, the non-wandering set consists of two non-trivial basic pieces and that it satisfies the strong transversality condition. Moreover, $Z^0(f \times g) \supset Z^0(f) \times \{\text{id}\}$, thus it is not discrete. Then, Theorem 2 guarantees there exists a continuum of conjugacies C^0 -close to the identity between $f \times g$ and any C^1 -diffeomorphism $F \in C_{f \times g}$. By structural stability, Theorem 2 implies that the later holds for a C^1 -open neighborhood of the diffeomorphism $f \times g$. In contrast to Example 7.1, the non-wandering set is formed by one attractor and one repeller, both non-trivial. By Theorem 1, every C^1 -small perturbation of $f \times g$ has sensitivity to initial conditions.

Finally, observe that the set of points with expansiveness are dense in $\mathbb{S}^1 \times \mathbb{T}^2$ while the centralizer is not discrete. This shows that dynamics with a dense subset of expansive points may have non-discrete C^0 -centralizer and proves that Walters lemma is optimal (cf. Lemma 3.1).

The following example shows that C^0 and C^1 -centralizers can be both discrete but distinct for a locally C^1 -Baire generic set of C^1 -diffeomorphisms.

Example 7.3. Let f be a C^1 -Anosov diffeomorphism on the two torus \mathbb{T}^2 with a unique fixed point. Then there exists a C^1 -open neighborhood \mathcal{U} of $f \in \text{Diff}^1(\mathbb{T}^2)$ formed subset of Anosov diffeomorphisms, topologically conjugate to f . The C^0 -centralizer of f is discrete but non-trivial (it has two generators) cf. [24]. Since every C^1 -diffeomorphism $g \in \mathcal{U}$ is topologically conjugate to f then it follows from Theorem 2 that the C^0 -centralizer of g is discrete but non-trivial. Now, we recall that there exists a C^1 -Baire residual subset $\mathcal{R} \subset \mathcal{U}$ diffeomorphisms so that $Z^1(g)$ is trivial for every $g \in \mathcal{R}$ (cf. [2]). Thus there exists a C^1 -Baire residual subset $\mathcal{R}_1 \subset \mathcal{U}$ so that $Z^0(g) \supsetneq Z^1(g) = \{g^n : n \in \mathbb{Z}\}$ for every $g \in \mathcal{R}_1$. In particular the analogous statement of Smale's conjecture is no longer true for C^0 -centralizers.

We also derive a consequence for reversible dynamics.

Example 7.4. Given an involution $R : M \rightarrow M$ (ie. $R^2 = \text{id}$) let $\text{Homeo}_R(M)$ denote the set of homeomorphisms f so that $R \circ f = f^{-1} \circ R$. These are called R -reversible homeomorphisms. Notice that $f \in \text{Homeo}_R(M)$ if

and only if f^{-1} belongs to the conjugacy class of $f \in \text{Homeo}(M)$ and that $R \in \mathcal{H}_{f,f^{-1}}$. Consider the (non-empty) set $\{h \in \text{Homeo}(M) : h \circ f = f^{-1} \circ h\}$ of conjugacies between f and f^{-1} . By Theorem 2, the map $F_R : Z^0(f) \rightarrow \mathcal{H}_{f,f^{-1}}$ given by $F_R(\tilde{f}) = R \circ \tilde{f}$ is an homeomorphism. Thus, if $f \in \text{Homeo}_R(M)$ then $Z^0(f)$ is discrete if and only if there exists $\varepsilon > 0$ so that there is no conjugacy between f and f^{-1} , distinct from R that is ε - C^0 -close to R . Clearly, both properties hold for reversible Anosov diffeomorphisms. Moreover, we deduce that if $Z^0(f)$ is trivial then every conjugation between f and f^{-1} is of the form $R \circ f^n$ for some $n \in \mathbb{Z}$.

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